SOLVING AN INVERSE PROBLEM FOR THE POSITIONS OF HEAT SOURCES AND SINKS IN A PLANE

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The configurational characteristics are considered for a planar stationary temperature pattern induced by a mutually screened system of small heating and cooling components.

The two-dimensional stationary temperature distribution within a body can [1, 2] be described by the complex potential w(z) = u(x, y) + iv(x, y), where z = x + iy, u(x, y) is temperature; the vector $U = -\varkappa \operatorname{grad} u = \varkappa w'(z)$ ($\varkappa$ is thermal conductivity) is the heat-flux one.

In the local representation of the complex potential

$$w(z) = -q(2\pi\varkappa)^{-1}\ln(z-a) + [\ldots + c_{-1}(z-a)^{-1} + c_0 + c_1(z-a) + \ldots] \tag{1}$$

the term $-q(2\pi\varkappa)^{-1}\ln(z-a)^1$ defines a source (a; q) at point a of output q, while $c_{-1}(z-a)^{-1}$ is a doublet, with the other negative powers in the Lorant expansion being higher multiplets.

Let the stationary temperature distribution in the plane be maintained by a finite system of sources and sinks split up into nonintersecting groups by assignment to the complete set of all different power outputs $\{q_j\}$. Each q_j is put into correspondence with $P_j(z)$, the generating polynomial for the group. The roots are the complex coordinates of all the sources of output q_j. The complex potential for such a system is

$$w(z) = (2\pi\varkappa)^{-1}\sum_j(-q_j)\ln P_j(z). \tag{2}$$

We estimate the configuration characteristics for this system with certain matched constraints on the components.

We represent the heat-flux vector in the region of each singularity z_k as

$$U_k = U_{0,k} + \tilde{U}_k, \tag{3}$$

where $U_{0,k}$ is the inherent heat flux from the singularity and $\tilde{U}_k$ is that induced by all the other singularities (incoming flux).

It is evident that

$$\tilde{U}_k(z_k) = -(2\pi)^{-1}\sum_j{}' q_j\frac{P_j'(z_k)}{P_j(z_k)},\quad P'(z) = \frac{d}{dz}P(z), \tag{4}$$

where the prime to the summation means that the singular term is excluded in expanding the right side of (4) as simple fractions.

We consider this singularity system with the following constraints on the regularized heat flux $\tilde{U}_k$:

$$\tilde{U}_k(z_k) = b_k \tag{5}$$

(the given b_k prescribe the modes of thermal action of the entire system on the components, i.e., we arrive at an interaction problem). In [3-5], a special method was developed for examining a nonlinear algebraic-equation system of the type of (5). We give some results based on this approach.

I. All the $b_k = 0$. The system state is such that the incoming fluxes have minimal thermal effect on the region around each singularity. In general, each singularity acquires the

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scope for operating under the most undistorted conditions (optimal ones). Then system (5) can be reduced to a differential relation for the generative polynomials P_j:

$$\sum_j q_j^2 P_j'' \prod_{m\neq j} P_m + 2 \sum_{m,j;m\neq j} q_j q_m P_j' P_m' \prod_{r\neq j,m} P_r = 0. \tag{6}$$

It is characteristic that this relation rigidly relates the numbers $n_j = \deg P_j$

$$\Big(\sum_j q_j n_j\Big)^2 = \sum_j q_j^2 n_j, \tag{7}$$

i.e., the composition of the singularities with given outputs $\{q_j\}$ is in no way arbitrary.

For example, if the system contains only sources +q (generative polynomial P, degP = n) and sinks −q (polynomial Q, degQ = m), the number of singularities in the system is given by

$$(n-m)^2 = n + m, \tag{8}$$

i.e., n and m are successive triangular numbers of the form L(L + 1)/2.

The polynomials P and Q satisfy the bilinear equation

$$P''Q + PQ'' - 2P'Q' = 0, \tag{9}$$

when [3] P(z) and Q(z) are adjacent terms in the soluble recurrent polynomial sequence

$$\begin{gathered} P_0 = 1,\ P_1 = z + \text{const},\ P_2,\ P_3,\ \ldots \\ P_{N+1}' P_{N-1} - P_{N+1} P_{N-1}' = (2N+1) P_N^2. \end{gathered} \tag{10}$$

Then for the ±q systems with regularized gradient constraint $\tilde{U}_k(z_k) = 0$ we can write the general representations for the complex potential

$$w(z) = -(2\pi\varkappa)^{-1} q \ln P_N(z)/P_{N-1}(z) \tag{11}$$

and the temperature distribution

$$u = -(2\pi\varkappa)^{-1} q \operatorname{Re}[\ln P_N(z)/P_{N-1}(z)]. \tag{12}$$

At large distances from all the singularities, the system works as a singularity of output Nq, and the isotherms pass asymptotically into circles.

II. In the system containing sources +q (polynomial P, degP = n) and sinks −2q (polynomial Q, degQ = m), $b_k = 0$.

The solutions to the corresponding bilinear equation

$$P''Q + 4PQ'' - 4P'Q' = 0 \tag{13}$$

are [5] the pairs of polynomials $\{P_N, Q_N\}$ defined from the recurrent system

$$\begin{gathered} Q_N' Q_{N-1} - Q_N Q_{N-1}' = (3N-1) P_N, \\ P_{N+1}' P_N - P_{N+1} P_N' = (6N-1) Q_N^4 \end{gathered} \tag{14}$$

with the initial conditions $P_0 = Q_0 = 1$, $P_1 = z$; $N > 0$) or ($P_0 = Q_0 = 1$, $Q_{-1} = z$, $N > 0$); the two branches in the solutions in (14) correspond to two branches in the solutions to the Diophantine equation for the degrees of the polynomials:

$$(n - 2m)^2 = n + 4m. \tag{15}$$

III. All $b_k = b \neq 0$. This case is naturally interpreted as the superposition of a constant heat flux on the entire system. Against the background of such superposition, the regularized interaction between the singularities may be considered as a zero one. However, the form of (6) is substantially altered, and we have

$$\sum_j q_j^2 P_j'' \prod_{m\neq j} P_m + 2 \sum_{m,j;m\neq j} q_j q_m P_j' P_m' \prod_{r\neq j,m} P_r = 4\pi\bar{b} \sum_j q_j P_j \prod_{m\neq j} P_m. \tag{16}$$

For example, the modification of (9) is

$$P''Q + PQ'' - 2P'Q' = 4\pi\bar{b}(P'Q - PQ'). \tag{17}$$

The solutions to this equation are the pairs of polynomials $\{P_N, Q_N\}$, $\deg P_N = \deg Q_N = N(N+1)/2$ [5].

IV. The method can be extended to multiplets [4]. One can for example write out the solution for an oriented doublet system. With $\tilde{U}_k = 0$, this is closely related to the polynomial class of (14).

In general, the mathematical techniques needed to solve this class of problems correspond to a series of mathematical schemes in the theory of nonlinear waves and solutions. This feature is evidently related to some deeper unity in all integrable nonlinear problems in transport theory for continuous media and for interactions in discrete systems.

V. There are papers in which it is shown that one can replace the extended thermal inhomogeneities by point singularities in planar potential theory for a certain class of thermal problems. In [6] this was done by considering the corresponding linked asymptotic expansions for a certain system of hot and cold small inclusions.

We have here demonstrated an exact technique that may be of value in analyzing and planning such reductions for two-dimensional thermal conduction problems (thermoelasticity and thermal stability) or problems on diffusion in inhomogeneous media.

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